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James G Berryman

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COMPARISON OF UP-SCALING METHODS IN POROELASTICITY AND ITS GENERALIZATIONS

James G. Berryman [1]

ABSTRACT

Four methods of up-scaling coupled equations at the microscale to equations valid at the mesoscale and/or macroscale for fluid-saturated and partially saturated porous media will be discussed, compared, and contrasted. The four methods are: (1) effective medium theory, (2) mixture theory, (3) two-scale and multiscale homogenization, and (4) volume averaging. All these methods have advantages for some applications and disadvantages for others. For example, effective medium theory, mixture theory, and homogenization methods can all give formulas for coefficients in the up-scaled equations, whereas volume averaging methods give the form of the up-scaled equations but generally must be supplemented with physical arguments and/or data in order to determine the coefficients. Homogenization theory requires a great deal of mathematical insight from the user in order to choose appropriate scalings for use in the resulting power-law expansions, while volume averaging requires more physical insight to motivate the steps needed to find coefficients. Homogenization often is performed on periodic models, while volume averaging does not require any assumption of periodicity and can therefore be related very directly to laboratory and/or field measurements. Validity of the homogenization process is often limited to specific ranges of frequency – in order to justify the scaling hypotheses that must be made – and therefore cannot be used easily over wide ranges of frequency. However, volume averaging methods can quite easily be used for wide band data analysis. So, we learn from these comparisons that a researcher in the theory of poroelasticity and its generalizations needs to be conversant with two or more of these methods to solve problems generally.

Keywords: poroelasticity, effective medium theory, homogenization, up-scaling

INTRODUCTION

The earth is typically probed with seismic waves in the range $1-100$ Hz, with well-logging tools in the range $1-50$ kHz, and samples of the earth in the laboratory from $200-1000$ kHz. The pertinent wave speeds for water and typical solid earth materials like quartz are, respectively, 1.5 km/s and about 6.0 km/s. So the range of wavelengths of interest in the field can vary from as much as 60 to 6000 m in the field to as little as 1.5 to 7.5 mm in the laboratory. Clearly the main purpose of laboratory measurements of earth materials is generally to elucidate the physical mechanisms of wave propagation in the earth. But the differences in the pertinent length scales is so great that unusual care must be taken to perform proper interpretation of the results — taking into account all the inherent problems with up-scaling.

[1]Univ. California, Lawrence Livermore National Lab., P.O. Box 808 L-200, Livermore, CA 94551-9900 Telephone: 925-423-2905 FAX: 925-423-6907 E-mail: berryman1@llnl.gov.

In particular, since earth materials are notoriously heterogeneous, it is very important to have some means of studying the effects of these heterogeneities on waves. So up-scaling in earth sciences applications is not an academic issue, but often the main scientific issue for many important applications.

The most common approach to dealing with earth heterogeneity for seismic waves is to assume the earth is homogeneous locally, but composed of many layers (Ewing et al. 1957; Brekhovskikh 1980). This approach can be useful for applications to large scale earth imaging and earthquake analysis. But in matters where fluids in the earth important, such as oil and gas exploration, hydrology, etc., the elastic approximation is usually not good enough and must either be supplemented or replaced altogether with more appropriate choices of equations and analysis methods. Early examples of such analyses include Biot (1941), Frenkel (1944), Gassmann (1951), Biot (1956a), Biot (1956b), Biot (1962) — all works which then provided a strong foundation for modern poroelastic analysis.

Our goal here will be to give a brief accounting of some of the most important methods used to do up-scaling in poroelasticity and also multi-scale poroelasticity. The main methods we consider here are: (1) effective medium theory (Kuster and Toksöz 1974; Berryman 1992; Berryman and Berge 1996), (2) mixture theory (Bedford and Drumheller 1979; Drumheller and Bedford 1980; Bedford and Drumheller 1983; Bedford 1985), (3) two-scale and multiscale homogenization (Burridge and Keller 1981; Auriault and Boutin 1994; Auriault and Royer 2002; Auriault 2002), and (4) volume averaging (Pride et al. 1992; Pride and Berryman 1998; Berryman and Pride 1998; Whitaker 1999; Whitaker 2002; Wood et al. 2003). All these methods have advantages for some applications and disadvantages for others. In a short review, it will not be possible to cover everything that is important. But we will try to cover here those relationships among the methods that are usually not explored in other discussions. In particular, it is the author's attitude that these methods are not so much competitive as they are complementary. It will be one of our main goals then to clarify why and in what sense this is true.

EFFECTIVE MEDIUM THEORIES

Effective medium theories for heterogeneous media have the advantage that they can be used to obtain direct estimates of overall physical constants such as bulk modulus, shear modulus, density, etc. There has been a tremendous amount of work done along these lines in the last 40 years, and recent progress on both bounding methods and effective medium theories was summarized recently by Milton (2002). In the earth sciences, the most commonly used effective medium theory in elasticity is the one developed by Kuster and Toksöz (1974). This theory and a related one [usually called the Mori-Tanaka method (Benveniste 1987)] have been discussed and their limitations explored by Berryman and Berge (1996).

For heterogeneous poroelasticity, there have been fewer effective medium theories developed and we will mention only two here, as originally described by Berryman (1992). The two approaches are the CPA (Coherent Potential Approximation) and the DEM (Differential Effective Medium). We present just the results here. The derivations can be found the references.

Coherent Potential Approximation

Suppose the heterogeneous medium is composed of some number (say n) of constituents, then the bulk modulus $K(\mathbf{x}) = K_j$, when $\mathbf{x}$ is inside the jth constituent, and similarly for the shear modulus G. Brackets $\langle\cdot\rangle$ imply a simple volume average. Then, for the bulk modulus

estimate of the CPA, we have

$$\frac{1}{K^*_{CPA}+\frac{4}{3}G^*_{CPA}}=\left\langle\frac{1}{K(\mathbf{x})+\frac{4}{3}G^*_{CPA}}\right\rangle, \tag{1}$$

while, for the shear modulus estimate, we have

$$\frac{1}{G^*_{CPA}+F^*_{CPA}}=\left\langle\frac{1}{G(\mathbf{x})+F^*_{CPA}}\right\rangle, \tag{2}$$

where the function $F=F(K,G)$ is defined by

$$F=(G/6)(9K+8G)/(K+2G). \tag{3}$$

Starred quantities in these formulas are the overall estimates, or else the functions evaluated at those same starred values. Note that (1) and (2) are implicit coupled equations, requiring simultaneous iteration for their solution. Formulas like (1) and (2) are based on the assumption that the inclusions are spherical in shape. For ellipsoidal shapes, (1) and (2) are modified based on Eshelby's well-known results in elasticity (Eshelby 1957; Mavko et al. 1998).

Differential Effective Medium

If there are only two constituents (a binary mixture) whose volume fractions are $x=v_1$ and $y=v_2=1-x$, then suppose we know the value of the effective bulk modulus $K^*_{DEM}(y)$ at one value of y. Treating $K^*_{DEM}(y)$ as the host medium and $K^*_{DEM}(y+dy)$ as the effective constant after a small proportion $dy/(1-y)$ of the host has been replaced by spherical inclusions of type-2, the bulk modulus estimate from the DEM method is determined by

$$\frac{K^*_{DEM}(y+dy)-K^*_{DEM}(y)}{K^*_{DEM}(y+dy)+\frac{4}{3}G^*_{DEM}(y)}=\frac{dy}{(1-y)}\frac{K_2-K^*_{DEM}(y)}{K_2+\frac{4}{3}G^*_{DEM}(y)}. \tag{4}$$

Since the host contains the volume fraction x of type-1 and y of type-2, on average a fraction $dy/(1-y)$ of the host must be replaced by type-2 in order to change the overall fraction of type-2 to $y+dy$. Taking the limit $dy\to 0$ gives the first order differential equation

$$(1-y)\frac{d}{dy}\left[K^*_{DEM}(y)\right]=\frac{K_2-K^*_{DEM}(y)}{K_2+\frac{4}{3}G^*_{DEM}(y)}\left[K^*_{DEM}(y)+\frac{4}{3}G^*_{DEM}(y)\right], \tag{5}$$

where the initial host is pure type-1 so $K^*_{DEM}(0)=K_1$. The corresponding formula for the shear modulus is

$$(1-y)\frac{d}{dy}\left[G^*_{DEM}(y)\right]=\frac{G_2-G^*_{DEM}(y)}{G_2+F^*_{DEM}(y)}\left[G^*_{DEM}(y)+F^*_{DEM}(y)\right], \tag{6}$$

where F is again determined as in (3). Note that (5) and (6) are coupled and must therefore be integrated simultaneously. There are two distinct results implied by (5) and (6), one for type-1 as host and another, by interchanging roles of the constituents, for type-2 as host. In contrast, CPA produces only one estimate and is, therefore, sometimes called the "symmetric self-consistent" approach. DEM is obviously non-symmetric in this sense, depending explicitly on one constituent or the other playing host.

Brief Discussion

Although both of these sets of formulas are actually of the same form as would be found in the simpler theory of elasticity, the constants and results quoted here for K^* and G^* are all assumed to be the frame constants for porous constituents and for the porous frame overall. Furthermore, both types of EMT estimates are supplemented in poroelasticity by additional equations for the other coefficients in the full set of poroelastic equations. Space will not permit us to pursue this discussion any more here. But the main point to be made about these methods is this: They sometimes give explicit, or more typically implicit (to be iterated or integrated), formulas for all the coefficients in the equations. Results are approximate, but for the methods presented here, the results are always within the known rigorous bounds (Milton 1985; Norris 1985; Avellaneda 1987; Milton 2002), and therefore always give reasonable approximations to the results when the form of the equations is known and only the values of the coefficients are needed. Related effective medium theory estimation methods have also been used recently to clarify the behavior of the poroelastic shear modulus in the presence of undrained fluids by Berryman et al. (2002).

MIXTURE THEORY

The term "mixture theory" is sometimes used as a generic term for any and all up-scaling methods. In this sense the venerated estimates of Voigt (1928) and Reuss (1929), which — for isotropic composites — are just the mean and harmonic mean respectively of the elastic constants, have been available since the 1920s and were for a long period of time about the only types of estimates known, except for effective medium theory estimates (Bruggeman 1935) similar in spirit to those described in the previous section. But Hill (1952) later showed that these estimates of Voigt and Reuss were really rigorous bounds on the elastic constants, thus changing our view of such estimates forever.

For our present purposes, we will use "mixture theory" in a different sense, motivated largely by work in the 1970's and 80's by Bedford and Drumheller (Bedford and Drumheller 1979; Drumheller and Bedford 1980; Bedford and Drumheller 1983; Bedford 1985). Their approach was based on energy principles, and all of this work could be viewed as variational methods using either Lagrange's or Hamilton's principles. In this regard, their work is in the same class as that of Biot's works on poroelasticity (Biot 1941; Biot 1956a; Biot 1956b; Biot 1962), as well as Berryman and Thigpen (1985), Lopatnikov and Cheng (2002), and — of course — many others. In their series of papers, Bedford and Drumheller showed how to use variational principles for up-scaling in a number of different complex systems, including but not limited to poroelastic systems.

One of the classic problems that must be dealt with in a poroelastic system — thus making the theory ultimately much more difficult conceptually than simple elastic or viscoelastic systems — is the fact that finite fluid permeability is an essential feature of all these systems. Without this feature, the system is just elastic or viscoelastic. Furthermore, the main poroelastic complications have to do with the ability of the fluid to move in or out of any given domain of the pore system, and the time scales over which this occurs. This capability is distinct from features of any elastic system, linear or nonlinear, fractured or unfractured, because the concept of a domain is itself rooted in the idea of the elastic or viscoelastic materials being in welded contact almost everywhere. Although any solid particle may move substantially from its original position at the beginning of any motion (or calculation), nevertheless it always remains relatively close to the same particles who were its neighbors at the beginning of the motion/calculation. This trait is not necessarily shared by fluid particles (although for very

small strains, as in the case of small amplitude sound waves, it is often satisfactory to treat the system if it is also true for the pore fluids). So for very small strains, it may be adequate in some cases to treat fluids purely elastically. But this case is probably again limited to problems that lie outside the domain of interest when we are studying poroelasticity. Circumstances resulting in trapped fluids can come into play dynamically if the frequencies of the motion involved are are very high (Kaelin and Johnson 1998), or if permeability values can fluctuate greatly during the motion so parts of the system become effectively undrained.

Biot's theories

Biot's papers on linear wave propagation in porous media (Biot 1956a; Biot 1956b; Biot 1962) all were based on a Lagrangian formulation that permitted the introduction of a dissipation functional in order to incorporate viscous losses due to the motion of pore liquids during the passage of the wave. The early pair of papers is probably best known to most readers, but the 1962 paper has conceptual advantages over the earlier ones, and so we tend to base our work more on this paper than on the other two — however, there is still much to be learned from all three. Biot also changed his notation for the coefficients in 1962, as this was made necessary by the introduction of the concept of the increment of fluid content ζ as one of the fundamental variables of the theory.

Energy functional

The reason for this shift is fairly easy to understand from the point of view Hamiltonian dynamics. There is no particular reason to prefer one choice of variables over another from the point of view of the kinetic energy in the system. Each physically sensible choice involves about the same analysis and the same amount of work. But, from the point of the internal energy of the macroscopic system there is a clear perference for the notation of Biot (1962). The reason is that, after we have analyzed the energy and macroscopic variables that go into it, the result for an isotropic system can be written (notation used here is not Biot's) as:

$$2E = \left[p_c^2 - 2\alpha p_c p_f + \alpha p_f^2/B\right]/K_d, \tag{7}$$

where p_c and p_f are, respectively, the external confining pressure and the fluid pore pressure, K_d is the bulk modulus of the drained system, α is the Biot-Willis parameter (Biot and Willis 1957), and B is Skempton's coefficient (Skempton 1954). All the variables and coefficients in this equation are well-defined macroscopic quantities that can be measured with relative ease in the laboratory. No microstructural variables appear here. Starting from this equation it is then natural in the context of Hamiltonian dynamics to ask what are the variables conjugate to the macroscopic stresses $\sigma_c = -p_c$ and $\sigma_f = -p_f$. The results are

$$e = \frac{\partial E}{\partial \sigma_c} \qquad \text{and} \qquad -\zeta = \frac{\partial E}{\partial \sigma_f}. \tag{8}$$

The strain e is simply the overall strain of the system. The new variable (from the point of view of elasticity) is the increment of fluid content $\zeta = \phi(e - e_f)$, where ϕ is the overall porosity of the system and e_f is the overall strain of the pore fluid.

The internal energy of the system can now be rewritten in terms of e and ζ as

$$2E = K_u \left[e^2 - 2Be\zeta + B\zeta^2/\alpha\right]. \tag{9}$$

The new coefficient appearing here is the undrained modulus of the system $K_u = K_d/(1 - \alpha B)$, also known as Gassmann's modulus (Gassmann 1951) — or the fluid substitution formula

(since the fluid bulk modulus is contained in B). The constitutive relations following from these energy equations are

$$e = -\left(p_c - \alpha p_f\right)/K_d \qquad \text{and} \qquad -\zeta = -\alpha\left(p_c - p_f/B\right)/K_d, \tag{10}$$

and

$$-p_c = K_u\left(e - B\zeta\right) \qquad \text{and} \qquad -p_f = BK_u\left(e - \zeta/\alpha\right). \tag{11}$$

The meaning of the coefficient K_d is very easy to understand as it is simply the bulk modulus of the drained system (liquid either absent or free to move in and out of the system as required under changes of confining pressure p_c, while p_f remains unchanged). Skempton's coefficient is defined for the other extreme of a fully undrained system, where the pore fluid pressure normally increases due to an increase in confirning pressure; thus, $B \equiv (\partial p_f/\partial p_c)|_{\zeta=0}$. Similarly, the Biot-Willis effective stress coefficient $\alpha \equiv (\partial p_c/\partial p_f)|_{e=0} = 1 - K_d/K_s$, where K_s is the "unjacketed modulus" of the porous system. The modulus K_s is exactly equal to the grain modulus if this is constant throughout the system (Brown and Korringa 1975), but its relationship to the moduli of the constituents is more complicated when multiple constituents are present. In general it should be treated as an effective modulus (Berryman and Milton 1991), or as an experimental parameter to be determined.

Wave equations of single-porosity poroelasticity

For long-wavelength disturbances ($\lambda >> h$, where h is a typical pore size) propagating through a single-porosity porous medium, we define average values of the (local) displacements in the solid and also in the saturating fluid. The average displacement vector for the solid frame is $\mathbf{u}$, while that for the pore fluid is $\mathbf{u}_f$. The average displacement of the fluid relative to the frame is $\mathbf{w} = \phi(\mathbf{u} - \mathbf{u}_f)$. For small strains, the frame dilatation is e, while the increment of fluid content is defined by

$$\zeta = -\nabla \cdot \mathbf{w} = \phi(e - e_f). \tag{12}$$

Biot introduces a kinetic energy functional in addition to the internal energy of (7) and then uses a standard Lagrangian variational approach to derive the wave equations of motion. With time dependence of the form $\exp(-i\omega t)$, the coupled wave equations that follow in the presence of dissipation are

$$\begin{aligned} -\omega^2(\rho\mathbf{u} + \rho_f\mathbf{w}) &= H\nabla e - C\nabla\zeta + G_d\left(\nabla^2\mathbf{u} - \nabla e\right), \\ -\omega^2(\rho_f\mathbf{u} + q\mathbf{w}) &= C\nabla e - M\nabla\zeta = -\nabla p_f, \end{aligned} \tag{13}$$

where G_d is the drained shear modulus, H, C, and M are bulk moduli,

$$\rho = \phi\rho_f + (1-\phi)\rho_m, \tag{14}$$

and

$$q(\omega) = \rho_f\left[\alpha/\phi + i\mathbf{F}(\xi)\eta/\kappa\omega\right]. \tag{15}$$

The kinematic viscosity of the liquid is η; the permeability of the porous frame is κ; the dynamic viscosity factor is given approximately [or see Johnson et al. (1987) for more discussion], for our choice of sign for the frequency dependence, by

$$\mathbf{F}(\xi) = \tfrac{1}{4}\{\xi T(\xi)/[1 + 2T(\xi)/i\xi]\}, \tag{16}$$

where

$$T(\xi) = \frac{\text{ber}'(\xi) - i\text{bei}'(\xi)}{\text{ber}'(\xi) - i\text{bei}'(\xi)} \tag{17}$$

and

$$\xi \equiv (\omega/\omega_0)^{\frac{1}{2}} = (\omega\alpha\kappa/\eta\phi)^{\frac{1}{2}} = (\omega h^2/\eta)^{\frac{1}{2}}. \tag{18}$$

The functions ber(ξ) and bei(ξ) are the real and imaginary parts of the Kelvin function. The dynamic parameter h is a characteristic length generally associated with and comparable in magnitude to the steady-flow hydraulic radius. The tortuosity $\alpha \geq 1$ is a pure number related to the frame inertia which has been measured (Johnson et al. 1982) and has also been estimated theoretically (Berryman 1980; Berryman 1983).

The coefficients H, C, and M are given by (Gassmann 1951; Geertsma 1957; Biot and Willis 1957; Geertsma and Smit 1961; Stoll 1974)

$$H = K_u + \frac{4}{3}G_d = K_d + \frac{4}{3}G_d + (1 - K_d/K_s)^2 M, \tag{19}$$

$$C = (1 - K_d/K_s)M, \tag{20}$$

where

$$M = 1/[(1 - \phi - K_d/K_s)/K_s + \phi/K_f]. \tag{21}$$

The constants are drained bulk and shear moduli K_d and G_d, the unjacketed bulk modulus K_s, and fluid bulk modulus K_f. Korringa (1981) showed equations (19)-(21) to be correct as long as the porous material may be considered homogeneous on the microscopic scale as well as the macroscopic scale. Also, see a recent tutorial on Gassmann's equations (Gassmann 1951) by Berryman (1999).

This set of equations together with the identification of the factors that determine the coefficients in the equations is the lasting contribution to the field of poroelasticity that we attribute to Biot, Gassmann, Skempton, and Willis by using their names in reference to the various equations and coefficients.

Up-scaling

At this point, the question could reasonably be asked "Has any up-scaling occurred yet?" Biot's work is very well motivated by the physics and by those quantities that can be measured and/or controlled in the laboratory. The equations shown are completely general for internal energies E of any poroelastic system. They may not provide everything we need to know or understand about how fluids move, or how energy is dissipated in these systems, but they are correct as far as they go. This theory is often described as "phenomenological," meaning that it provides equations that describe the behavior of the physical system of interest, but still requires measurements for the coefficients to be known. This characterization is accurate, but it should not be interpreted as an implicit criticism of the theory. To take a critical point of view consistently for all such phenomenological theories is surely not fruitful, as for example most theories including those of elasticity, Maxwell's equations, etc., are also phenomenological in exactly the same sense.

What Biot has accomplished is very important nevertheless, as these equations are sufficiently general that we can start making use of them to do up-scaling. In particular, because of their very general nature, the same equations apply both at the microscale and the macroscale, as well as at all the mesoscales in between these limits. There is a lower limit below which Biot's theory cannot be used, and this comes in the form of an REV (representative elementary volume). This volume needs to have sufficient size so that it will generally include both solid grains and fluids simultaneously. A typical rule of thumb is that a cubical REV should be about two to five grains on a side (Bourbié et al. 1987), or more. The optimum size of the REV can depend on the wavelength of the waves, but for seismic waves with large wavelengths this is seldom an issue. For larger REVs and longer wavelengths, it clearly becomes essential to consider up-scaling as the required effective coefficients in Biot's equations may contain a great many types of grains, pores, and fluids — and at many scales.

Mixture theory becomes relevant in this context because it is based on energy concepts, much like Biot's approach. However, mixture theory starts at the level of the individual constituents, keeping track of the energy in each of these, and then providing a means of generating the equations of motion for the overall system. In one of their first applications of this method, Bedford and Drumheller (1979) applied their technique to porous media. The framework of their approach was Eulerian, which is sensible for porous media when the resulting equations of motion are linear. Berryman and Thigpen (1985) applied the same basic method to nonlinear and especially semilinear [as defined by Biot (1973)] porous systems, and showed that a Lagrangian approach was preferred in this case. But the resulting linear equations of motion are the same for either the Eulerian or Lagrangian methods. The details of this work are a bit too specialized for the present short review, and we will leave it to the interested reader to follow up with the pertinent references.

HOMOGENIZATION THEORY

A two-space method of homogenization leading to equations having the form of Biot's equations has been presented by Burridge and Keller (1981). This method has been developed by various authors including Bensoussan et al. (1978), Keller (1977), and Sanchez-Palencia (1980). The method requires that the microscale of the heterogeneous porous medium is much smaller than the macroscale of most interest. The method is systematic, leading to equations at the macroscale from an analysis of the microscale behavior, which for the present problem involves assuming the the solid components obey linearized equations of elasticity, while the fluid components obey linearized Navier-Stokes equations. Burridge and Keller (1981) show that there are actually two possible solutions to the problem. One solution is essentially that of Biot's theory of wave propagation in poroelastic media. The other is a set of viscoelastic equations. The small quantity ϵ, being the ratio of the microscale size to the macroscale size, is used to characterize various scaling regimes. The difference leading to the two quite different results found by Burridge and Keller is that, when the scaled viscosity is treated as being of order ϵ^2, they get the Biot-Gassmann equations, whereas when it is treated as order unity, they obtain equations of viscoelasticity instead. In the language of poroelasticity, the case leading to viscoelastic equations is what is normally termed "undrained," meaning that the fluid does not have sufficient time for its pressure to equilibrate at the microscale throughout the macromedium on the time scales of interest. This failure to equilibrate can occur due to low fluid permeabilities, high viscosity, very high wave frequencies, or combinations of all these effects when present.

The approach involves assuming that any quantity Q can be treated as if it is a function of

the two spatial scales $\mathbf{x}$ and $\mathbf{y} = \mathbf{x}/\epsilon$. The macroscale is $\mathbf{x}$ and the microscale is $\mathbf{y}$. Spatial gradients ∇ of Q can then be usefully written as

$$\nabla Q(\mathbf{x}, \mathbf{y}) = \nabla Q(\mathbf{x}, \mathbf{x}/\epsilon) = \nabla_{\mathbf{x}} Q + \epsilon^{-1} \nabla_{\mathbf{y}} Q. \tag{22}$$

Thus, the scale separation can be explicitly and simply accounted for in such gradient equations. Furthermore, each quantity Q can also be treated as a function of ϵ, so that an asymptotic expansion of the form

$$Q(\mathbf{x}, \mathbf{y}, \epsilon) = Q_0(\mathbf{x}, \mathbf{y}) + \epsilon Q_1(\mathbf{x}, \mathbf{y}) + \frac{\epsilon^2}{2} Q_2(\mathbf{x}, \mathbf{y}) + O(\epsilon^2) \tag{23}$$

may be written. Combining (22) and (23) gives

$$\nabla Q = \epsilon^{-1} \nabla_{\mathbf{y}} Q_0(\mathbf{x}, \mathbf{y}) + [\nabla_{\mathbf{x}} Q_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} Q_1(\mathbf{x}, \mathbf{y})] + O(\epsilon), \tag{24}$$

a result which gets used repeatedly in the subsequent analysis. Furthermore, Eq. (24) already suggests the important result that, when ϵ is small — *i.e.*, tending to zero, it must generally be true that

$$\nabla_{\mathbf{y}} Q_0(\mathbf{x}, \mathbf{y}) = 0, \tag{25}$$

which is in fact a common result of this analysis.

If we let Ω_s be the domain occupied by solid, Ω_f the domain occupied by fluid, and ∂I_{sf} be the interface between solid and fluid, then the linearized equations for elasticity of the solid in Ω_s are

$$-\omega^2 \rho_s \mathbf{u}_s = \nabla \cdot \tau, \quad \text{where} \quad \tau = L \nabla \mathbf{u}_s, \tag{26}$$

the linearized equations of Navier-Stokes for the fluid are

$$i\omega \rho_f \mathbf{v}_f = \nabla \cdot \sigma_f \quad \text{where} \quad \sigma_f = -p_f I + \nu D \nabla \mathbf{v}_f \quad \text{and} \quad i\omega p_f = -\nabla \cdot \mathbf{v}_f / K_f. \tag{27}$$

The boundary conditions at the interfaces ∂I_{sf} are no slip: $\mathbf{v}_f = i\omega \mathbf{u}_s$, and continuity of normal stress: $\mathbf{n} \cdot \sigma_f = \mathbf{n} \cdot \tau$. The fluid and solid densities are ρ_f and ρ_s, respectively. The fluid viscosity is ν, and its bulk modulus is K_f. The stress tensors for fluid and solid are σ_f and τ, respectively, and p_f is the fluid pressure. L is the fourth rank elastic stiffness tensor, and D is the operator that produces the symmetrized deviatoric part of a second rank tensor.

We will use a notation slightly different from that of Burridge and Keller (1981) in order to facilitate the comparisons between these results and those of Biot. Space constraints will not permit us to follow the derivation of the equations further here. But one of the final macroscale results of the analysis is given by

$$-\omega^2 (\rho \mathbf{u}_0 + \rho_f \overline{\mathbf{w}}) = \nabla_{\mathbf{x}} \cdot (\overline{\tau_0} - \phi p_0 I), \tag{28}$$

where $\rho = (1 - \phi)\rho_s + \phi \rho_f$, and ϕ is the porosity. The overbar indicates a volume average over the fast variable $\mathbf{y}$. The second macroscale result is

$$-\omega^2 [\rho_f \mathbf{u}_0 + \Gamma(\omega) \overline{\mathbf{w}}] = -\nabla_{\mathbf{x}} p_0. \tag{29}$$

where $\Gamma(\omega)$ is a viscodynamic operator. The theory also shows that the macroscale stress and fluid pressure are determined by

$$\overline{\tau_0} - \phi p_0 I = J \nabla_{\mathbf{x}} \mathbf{u}_0 + C \nabla_{\mathbf{x}} \cdot \overline{\mathbf{w}} I \tag{30}$$

and

$$p_0 = -C \nabla_{\mathbf{x}} \cdot \mathbf{u}_0 - M \nabla_{\mathbf{x}} \cdot \overline{\mathbf{w}}, \tag{31}$$

where C and M are well defined scalar coefficients and J is a fourth rank tensor, all of which arise naturally within the two-scale analysis.

When equations (28)–(31) are compared with Biot's equations (13), we find that the form of these equations is identical — once we have taken care to interpret each of these expressions in terms of the corresponding expressions in the other set of equations, as was done in the original publication by Burridge and Keller (1981). Thus, the two-space homogenization method produces exactly the same equations as Biot found using his variational approach. One advantage that the present method has is that it also produces definite formulas for the coefficients in these equations, so — at least in principle — model calculations can be done to produce a set of theoretical examples to study the quantitative behavior of these coefficients. As far as I am aware, this step has never been taken. It is not necessarily easy to compute these coefficients from the formulas, but it would nevertheless be an interesting exercise in the theory to do so.

In contrast, the volume averaging methods to be discussed next also produce the same equations, but they do not produce formulas for the coefficients. So the volume averaging approach is phenomenological, *i.e.*, producing a set of equations whose coefficients must be determined experimentally.

VOLUME AVERAGING METHODS

Pride et al. (1992) studied the way in which the equations of motion for sound traveling through a solid/fluid mixture can be derived from first principles when it is assumed that the solid is porous, but contains only a single type of mineral. The fluid is homogeneous and completely fills the pores. Various other authors have also studied volume averaging both for the simple single-constituent poroelasticity and for multi-constituent generalizations such as double-porosity poroelasticity (Tuncay and Corapcioglu 1995; Pride and Berryman 1998; Berryman and Pride 1998; Pride and Berryman 2003a; Pride and Berryman 2003b).

The averaging theorem

The averaging theorem used by all these authors is due to Slattery (1967) and is based on well-known mathematics (Green's theorem and the divergence theorem) together with the idea that in relatively small regions *volume averages of spatial gradients* in statistically homogeneous media are presumably closely related to *gradients of volume averages*. But care must nevertheless be taken to account properly for behavior of the averaged quantities at points or surfaces where abrupt changes occur. In particular, when the quantity to be averaged exists on one side of an interface and does not exist on the other side, an interior interface term will contribute to the volume average of the derivative, but not to the derivative of the volume average.

Suppose that Q is a quantity to be averaged. Q can be a scalar, vector, or tensor. For convenience of the discussion, we will assume that the averaging volume is a finite sphere centered at position $\mathbf{x}$, although other choices are also possible (Pride and Berryman 1998). We label this volume $\Omega(\mathbf{x})$ and the surface of this volume is $\partial\Omega$. The exterior surface has two parts

$\partial\Omega = \partial E_0 + \partial E_Q$, with ∂E_0 being the part where the quantity of interest Q vanishes identically and ∂E_Q being the part where $Q \neq 0$. For example, Q could represent some physical quantity in the pore space and 0 in the solid — or vice versa — depending on immediate interest. In addition to the exterior surface, there are also interior surfaces where Q changes abruptly to zero and we label such surfaces ∂I_Q, for interior. The interior surface is the bounding surface for the region we will label Ω_Q, *i.e.*, the region wherein the quantity Q to be averaged is nonzero. With these definitions, Green's theorem gives

$$\int_\Omega \nabla Q \, d^3x = \int_{\Omega_Q} \nabla Q \, d^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q \, dS + \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, dS, \tag{32}$$

where dS is the infinitesimal of the surface volume element, and $\hat{\mathbf{n}}_Q$ is the unit outward normal vector from the region containing nonzero Q. The main point of (32) is just that $\partial E_Q + \partial I_Q$ is the entire bounding surface of Q in the volume Ω. As an example of the meaning of this result, consider Q to be a vector quantity, take the trace of (32), and the result is just a statement of the well-known divergence theorem for vectors.

A second result of interest is that

$$\nabla \int_\Omega Q \, d^3x = \nabla \int_{\Omega_Q} Q \, d^3x = \int_{\partial E_Q} \hat{\mathbf{n}}_Q Q \, dS. \tag{33}$$

The result (33) follows from the fact that the volumes $\Omega(\mathbf{x})$ and $\Omega(\mathbf{x} + \delta\mathbf{x})$ contain virtually the same internal surfaces (in the limit $\delta\mathbf{x} \to 0$ they are obviously identical) and so these do not contribute to the gradient.

Combining these results finally gives

$$\int_{\partial E_Q} \hat{\mathbf{n}}_Q Q \, dS = \nabla \int_\Omega Q \, d^3x = \int_\Omega \nabla Q \, d^3x - \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, dS. \tag{34}$$

Dividing by the total volume $V = \int_\Omega d^3x$ (which is a constant scalar, since the size of Ω is the same everywhere) contained in Ω gives the averaging theorem:

$$\nabla \langle Q \rangle = \langle \nabla Q \rangle - \frac{1}{V} \int_{\partial I_Q} \hat{\mathbf{n}}_Q Q \, dS. \tag{35}$$

Also note that the average $\langle Q \rangle$ is an average over the whole volume of Ω, while we also sometimes need to consider the partial average $\bar{Q}$, related to the full volume average by

$$\langle Q \rangle = \bar{v}_Q \bar{Q}, \tag{36}$$

where $\bar{v}_Q$ is the volume fraction of Ω in which Q is nonzero.

Finally, although this dependence is often not explicitly shown or even mentioned, all the average quantities are in fact functions of the particular choice of averaging volume $\Omega(\mathbf{x})$. In principle, $\Omega(\mathbf{x})$ can be as large as the sample being studied, or as small as desired. The legitimacy of the averaging theorem itself does not depend at all on the size of this averaging volume. However, the usefulness of the resulting meso- or macro-scale equations does depend on this choice and so some intermediate size is generally picked for $\Omega(\mathbf{x})$. Too small of an averaging volume implies rapid fluctuations in the quantities of interest (like the fluid and solid dilatations), while a very large averaging volume implies all the coefficients in the equations are universal constants and, therefore, can prevent us from studying the effects of local inhomogeneities, whenever they are present.

Note, for example, that a most desirable (but not always correct) consequence of (35) is for the final surface integral to vanish identically. The vanishing of this integral is natural in statistically homogeneous media because the unit outward normal vector averages to zero if Q is approximately constant on this surface. Vanishing of this surface integral is therefore often highly likely in reasonably homogeneous media in 3D (averaging over a 2D surface), still likely but somewhat less so in 2D (averaging over a 1D curve), and in general will not vanish in 1D (averaging over just two points) for any but some rather trivial model problems. So volume averaging methods should be replaced in 1D by exact methods such as, for example, Backus averaging (Backus 1962) for pertinent 1D applications.

In wave problems, when Ω is larger than the wavelength, the displacements will tend to average to zero, which is clearly an undesirable result. Pride *et al.* (1992) provide further discussion of criteria for choosing the size of the averaging volume. Thus, the choice of the averaging volume is often based on the same or similar issues normally used to pick an REV (representative elementary volume) in other methods, but we believe it is useful to maintain a strict distinction between these two concepts as the motivations for choices made are sometimes different.

Applications

Volume averaging has been applied successfully to derive the form of Biot's equations of poroelasticity (Pride et al. 1992), and more recetly a wide variety of other up-scaling problems in double-porosity poroelasticity (Tuncay and Corapcioglu 1995; Pride and Berryman 1998; Berryman and Pride 1998; Pride and Berryman 2003a; Pride and Berryman 2003b). The method is well-suited to obtaining the forms of the equations, but needs to be supplemented when the values of the coefficients in the equations are required. The supplements can obviously be obtained experimentally, in which case the theory can be treated as a phenomenological one — like Biot's original formulation using Lagrangian variational principles. But, as explained previously, being phenomenological is not a major limitation since of most of the theories and equations of mathematical physics are in fact phenomenological in the same sense. There are some cases in poroelasticity where various other theoretical means, including some of those already mentioned here, such as effective medium theories and periodic cell homogenization theory, can be applied to obtain estimates of the constants (Mavko et al. 1998; Milton 2002). And in some special cases, exact results are known (Berryman and Milton 1991; Berryman and Pride 2002) for a two-component solid matrix. In these situations the problems can be solved explicitly and quite easily. In most other situations, it remains an open question whether the coefficients in the equations can be determined accurately either by exact or some well-controlled but approximate means.

CONCLUSIONS

Four methods of up-scaling coupled equations at the microscale to equations valid at the mesoscale and/or macroscale for fluid-saturated and partially saturated porous media have been discussed, compared, and contrasted. The four methods were: (1) effective medium theory, (2) mixture theory, (3) two-scale and multiscale homogenization, and (4) volume averaging. All these methods have advantages for some applications and disadvantages for others. Effective medium theory, mixture theory, and homogenization methods can all give formulas for coefficients in the up-scaled equations, whereas volume averaging methods give the form of the up-scaled equations but generally must be supplemented with physical arguments and/or data in order to determine the coefficients. Homogenization theory requires a great deal of mathematical insight from the user in order to choose appropriate scalings for use in the resulting

power-law expansions, while volume averaging requires more physical insight to motivate the steps needed to find coefficients. Homogenization often is performed on periodic models, while volume averaging does not require any assumption of periodicity and can therefore be related very directly to laboratory and/or field measurements. Validity of the homogenization process is often limited to specific ranges of frequency – in order to justify the scaling hypotheses that are made – and therefore cannot be used easily over wide ranges of frequency. However, volume averaging methods can quite easily be used for wide band data analysis. So, we learn from these comparisons that a researcher in the theory of poroelasticity and its generalizations needs to be conversant with more than one of the methods to solve problems generally.

In this short review, we have not attempted to cover all methods that might be of interest and value for the applications considered. In particular, we have avoided discussion of ensemble averaging methods as well as other methods that might take the details of the spatial statistics of the complex heterogeneous media directly into account, or provide additional information about important corrections to the average equations. Recent publications by Drugan and Willis (1996) and Drugan (2003) suggest that such methods may also be of great value for analysis of the effective equations of poroelasticity, but these applications await further development.

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